

A HOM-ASSOCIATIVE ANALOGUE OF n -ARY HOM-NAMBU ALGEBRAS

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ABSTRACT. It is shown that every n -ary totally Hom-associative algebra with equal twisting maps yields an n -ary Hom-Nambu algebra via an n -ary version of the commutator bracket. The class of n -ary totally Hom-associative algebras is shown to be closed under twisting by self-weak morphisms. Every multiplicative n -ary totally Hom-associative algebra yields a sequence of multiplicative totally Hom-associative algebras of exponentially higher arities. Under suitable conditions, an n -ary totally Hom-associative algebra gives an $(n - k)$ -ary totally Hom-associative algebra.

1. INTRODUCTION

Nambu algebras play an important role in physics. For example, Nambu mechanics [16, 20] involves n -ary Nambu algebras [8, 18]. In these n -ary Nambu algebras, the n -ary compositions are derivations, a property called the n -ary Nambu identity (a.k.a. Filippov identity), which generalizes the Jacobi identity. Lie triple systems [10, 11], which are ternary Nambu algebras with two further properties, can be used to solve the Yang-Baxter equation [17]. An n -ary Nambu algebra whose product is anti-symmetric is called an n -ary Nambu-Lie algebra (a.k.a. Filippov algebra). Ternary Nambu-Lie algebras appear in the work of Bagger and Lambert [6] and many others on M-theory. Other applications of n -ary Nambu algebras in physics are discussed in, e.g., [4, 17, 18].

Generalizations of n -ary Nambu(-Lie) algebras, called n -ary Hom-Nambu(-Lie) algebras, were introduced by Ataguema, Makhlouf, and Silvestrov in [3]. In an n -ary Hom-Nambu(-Lie) algebra, the n -ary Nambu identity is replaced by the n -ary Hom-Nambu identity (see Definition 4.4), which involves $n - 1$ linear twisting maps. These twisting maps can be thought of as additional degrees of freedom or as deformation parameters. An n -ary Nambu(-Lie) algebra can be regarded as an n -ary Hom-Nambu(-Lie) algebra in which the twisting maps are all equal to the identity map. Hom-Nambu(-Lie) algebras are interesting even when the underlying algebras are Nambu(-Lie) algebras. In fact, certain ternary Nambu-Lie algebras can be regarded as ternary Hom-Nambu-Lie algebras with non-identity twisting maps [3].

Binary Hom-Nambu-Lie algebras are called Hom-Lie algebras, which originated in [9] in the study of q -deformations of the Witt and the Virasoro algebras. The binary Hom-Nambu identity in this case is called the Hom-Jacobi identity [14]. The associative counterparts of Hom-Lie algebras are Hom-associative algebras [14]. Hom-Lie algebras are to Hom-associative algebras as Lie algebras are to associative algebras [14, 21].

Let us recall some properties of n -ary Hom-Nambu(-Lie) algebras. It is shown in [3] that n -ary Nambu(-Lie) algebras can be twisted along self-morphisms to yield n -ary Hom-Nambu(-Lie) algebras. This twisting construction of n -ary Hom-Nambu(-Lie) algebras is a generalization of a result about G -Hom-associative algebras due to the author [22]. It is shown in [1] that ternary

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Virasoro-Witt algebras can be q -deformed into ternary Hom-Nambu-Lie algebras. It is shown in [2] that a ternary Hom-Nambu-Lie algebra can be obtained from a Hom-Lie algebra together with a compatible linear map and a trace function.

Further properties of n -ary Hom-Nambu(-Lie) algebras were established by the author in [24, 25]. A unique feature of Hom-type algebras is that they are closed under twisting by suitably defined self-morphisms. In particular, it is shown in [24] that the category of n -ary Hom-Nambu(-Lie) algebras is closed under twisting by self-weak morphisms. Starting with an n -ary Nambu(-Lie) algebra, this closure property reduces to the twisting construction for n -ary Hom-Nambu(-Lie) algebras in [3]. Moreover, it is proved in [24] that every multiplicative n -ary Hom-Nambu algebra yields a sequence of Hom-Nambu algebras of exponentially higher arities. It is also shown in [24] that, under suitable conditions, an n -ary Hom-Nambu(-Lie) algebra reduces to an $(n-k)$ -ary Hom-Nambu(-Lie) algebra.

Hom-Jordan and Hom-Lie triple systems were defined in [24] as Hom-type generalizations of Jordan and Lie triple systems [10, 11, 15]. A Hom-Lie triple system is automatically a ternary Hom-Nambu algebra, but it is usually not a ternary Hom-Nambu-Lie algebra because its ternary product is not assumed to be anti-symmetric. It is proved in [24] that Hom-Lie triple systems, and hence ternary Hom-Nambu algebras, can be obtained from Hom-Jordan triple systems, ternary totally Hom-associative algebras [3], multiplicative Hom-Lie algebras, and Hom-associative algebras.

Furthermore, it is proved in [25] that multiplicative Hom-Jordan algebras [23] have underlying Hom-Jordan triple systems, and hence also ternary Hom-Nambu algebras. Combined with results from [24], this implies that every multiplicative Hom-Jordan algebra gives rise to a sequence of Hom-Nambu algebras of arities $2^{k+1} + 1$. As in the classical case, a major source of Hom-Jordan algebras is the class of Hom-alternative algebras, which were defined in [13]. It is proved in [23] that multiplicative Hom-alternative algebras are Hom-Jordan admissible. Therefore, every multiplicative Hom-alternative algebra also gives rise to a sequence of Hom-Nambu algebras of arities $2^{k+1} + 1$. Finally, the class of n -ary Hom-Nambu-Lie algebras is extended to the class of n -ary Hom-Maltsev algebras in [25].

The main purpose of this paper is to study a Hom-associative analogue of n -ary Hom-Nambu algebras. The basic motivation is that the n -ary Hom-Nambu identity is an n -ary version of the Hom-Jacobi identity. As is well-known, the commutator bracket of an associative algebra satisfies the Jacobi identity. Likewise, the commutator bracket of a Hom-associative algebra satisfies the Hom-Jacobi identity [14]. Therefore, it is natural to ask the following question.

Is there an n -ary version of a Hom-associative algebra that gives rise to an n -ary Hom-Nambu algebra via an n -ary version of the commutator bracket?

One main result of this paper is an affirmative answer to this question when the twisting maps are equal (Theorem 4.5). The relevant Hom-associative type objects are the n -ary totally Hom-associative algebras defined in [3], which generalize n -ary totally associative algebras. The relevant commutator bracket is what we call the n -commutator bracket (Definition 4.3), which involves 2^{n-1} terms. Restricting to the special case where all the twisting maps are equal to the identity map, this result implies that every n -ary totally associative algebra yields an n -ary Nambu algebra via the n -commutator bracket.

Low-dimensional cases of the n -commutator bracket have been used elsewhere. In particular, the 2-commutator bracket is the usual commutator. The 3-commutator bracket was used by the author in [24] to show that a ternary totally Hom-associative algebra with equal twisting maps yields a

ternary Hom-Nambu algebra. We should point out that our n -commutator bracket is not the same as the totally anti-symmetrized n -ary commutator in [5, 7]. Moreover, the n -commutator bracket is not anti-symmetric when $n \geq 3$. Therefore, the n -ary Hom-Nambu algebras arising from n -ary totally Hom-associative algebras via the n -commutator bracket are usually not n -ary Hom-Nambu-Lie algebras.

A description of the rest of this paper follows.

In section 2 we observe that the class of n -ary totally Hom-associative algebras is closed under twisting by self-weak morphisms (Theorem 2.5). The corresponding closure property for n -ary Hom-Nambu(-Lie) and n -ary Hom-Maltsev algebras can be found in [24] and [25], respectively. A special case of Theorem 2.5 says that each multiplicative n -ary totally Hom-associative algebra gives rise to a sequence of multiplicative n -ary totally Hom-associative algebras by twisting along its own twisting map (Corollary 2.7). We obtain Theorem 3.6 in [3] as another special case of Theorem 2.5. It says that n -ary totally associative algebras can be twisted along self-morphisms to yield multiplicative n -ary totally Hom-associative algebras (Corollary 2.8). Section 2 ends with several examples of n -ary totally Hom-associative algebras.

In section 3 we study how totally Hom-associative algebras of different arities are related. First, we show that every multiplicative n -ary totally Hom-associative algebra yields a sequence of multiplicative totally Hom-associative algebras of exponentially higher arities, namely, $2^k(n-1)+1$ for $k \geq 0$ (Corollary 3.2). The transition from Hom-associative algebras to ternary totally Hom-associative algebras was proved in [24]. One major difference between the $n=2$ case in [24] and the $n \geq 3$ case here is that the latter requires multiplicativity while the former does not. Second, we show that under suitable conditions an n -ary totally Hom-associative algebra reduces to an $(n-k)$ -ary totally Hom-associative algebra (Corollary 3.5). The corresponding results for Hom-Nambu algebras can be found in [24].

In section 4 we define the n -commutator and show that every n -ary totally Hom-associative algebra with equal twisting maps yields an n -ary Hom-Nambu algebra via the n -commutator (Theorem 4.5). The special case of this result when $n=2$ is Proposition 1.6 in [14]. The special case when $n=3$ is Corollary 4.3 in [24]. In these low dimensional cases, due to the relatively small number of terms involved, the binary and ternary Hom-Nambu identities can be shown by a direct computation with all the terms written out. In the general case, a more systematic argument is needed because of the large number of terms in the Hom-Nambu identity. In fact, the n -ary Hom-Nambu identity for the n -commutator bracket involves $2^{2n-2}(n+1)$ terms.

2. TOTALLY HOM-ASSOCIATIVE ALGEBRAS

In this section we observe that the class of n -ary totally Hom-associative algebras is closed under twisting by self-weak morphisms. Some examples of n -ary totally Hom-associative algebras are then given.

2.1. Conventions. Throughout this paper we work over a fixed field \mathbf{k} of characteristic 0. If V is a \mathbf{k} -module and $f: V \rightarrow V$ is a linear map, then f^n denotes the composition of n copies of f with $f^0 = Id$. For $i \leq j$, elements $x_i, \dots, x_j \in V$ and maps $f, f_k, \dots, f_l: V \rightarrow V$ with $j-i = l-k$, we

adopt the abbreviations

$$\begin{aligned} x_{i,j} &= (x_i, x_{i+1}, \dots, x_j), \\ f(x_{i,j}) &= (f(x_i), f(x_{i+1}), \dots, f(x_j)), \\ f_{k,l}(x_{i,j}) &= (f_k(x_i), f_{k+1}(x_{i+1}), \dots, f_l(x_j)). \end{aligned} \tag{2.1.1}$$

For $i > j$, the symbols $x_{i,j}$, $f(x_{i,j})$, and $f_{k,l}(x_{i,j})$ denote the empty sequence. For a bilinear map $\mu: V^{\otimes 2} \rightarrow V$, we often write $\mu(x, y)$ as the juxtaposition xy .

Let us begin with the following basic definitions.

Definition 2.2. Let $n \geq 2$ be an integer.

- (1) An n -ary **Hom-algebra** $(V, (\dots), \alpha)$ with $\alpha = (\alpha_1, \dots, \alpha_{n-1})$ [3] consists of a \mathbf{k} -module V , an n -linear map $(\dots): V^{\otimes n} \rightarrow V$, and linear maps $\alpha_i: V \rightarrow V$ for $i = 1, \dots, n-1$, called the **twisting maps**.
- (2) An n -ary Hom-algebra $(V, (\dots), \alpha)$ is said to be **multiplicative** if (i) the twisting maps are all equal, i.e., $\alpha_1 = \dots = \alpha_{n-1} = \alpha$, and (ii) $\alpha \circ (\dots) = (\dots) \circ \alpha^{\otimes n}$.
- (3) A **weak morphism** $f: V \rightarrow U$ of n -ary Hom-algebras is a linear map of the underlying \mathbf{k} -modules such that $f \circ (\dots)_V = (\dots)_U \circ f^{\otimes n}$. A **morphism** of n -ary Hom-algebras is a weak morphism such that $f \circ (\alpha_i)_V = (\alpha_i)_U \circ f$ for $i = 1, \dots, n-1$ [3].

For an n -ary Hom-algebra V and elements $x_1, \dots, x_n \in V$, using the abbreviations in (2.1.1), the n -ary product (x_1, \dots, x_n) will often be denoted by $(x_{1,n})$ below. We sometimes omit the commas in the n -ary product (\dots) .

An n -ary Hom-algebra V in which all the twisting maps are equal, as in the multiplicative case, will be denoted by $(V, (\dots), \alpha)$, where α is the common value of the twisting maps. An n -ary **algebra** in the usual sense is a \mathbf{k} -module V with an n -linear map $(\dots): V^{\otimes n} \rightarrow V$. We consider an n -ary algebra $(V, (\dots))$ also as an n -ary Hom-algebra $(V, (\dots), Id)$ in which all $n-1$ twisting maps are the identity map. Also, in this case a weak morphism is the same thing as a morphism, which agrees with the usual definition of a morphism of n -ary algebras.

Let us now recall the definition of an n -ary totally Hom-associative algebra from [3].

Definition 2.3. Let $(A, (\dots), \alpha)$ be an n -ary Hom-algebra.

- (1) For $i \in \{1, \dots, n-1\}$ define the i th **Hom-associator** $as_A^i: A^{\otimes 2n-1} \rightarrow A$ to be the $(2n-1)$ -linear map

$$\begin{aligned} as_A^i(a_{1,2n-1}) &= (\alpha_{1,i-1}(a_{1,i-1}), (a_{i,i+n-1}), \alpha_{i,n-1}(a_{i+n,2n-1})) \\ &\quad - (\alpha_{1,i}(a_{1,i}), (a_{i+1,i+n}), \alpha_{i+1,n-1}(a_{i+n+1,2n-1})) \end{aligned}$$

for $a_1, \dots, a_{2n-1} \in A$.

- (2) An n -ary **totally Hom-associative algebra** is an n -ary Hom-algebra A that satisfies **total Hom-associativity**

$$as_A^i = 0$$

for all $i \in \{1, \dots, n-1\}$.

An n -ary totally Hom-associative algebra with $\alpha_i = Id$ for all i is called an n -ary **totally associative algebra**. In this case, $as_A^i = 0$ is referred to as **total associativity**.

When $n = 2$ total Hom-associativity means

$$((a_1 a_2) \alpha(a_2)) = (\alpha(a_1)(a_2 a_3)),$$

which is the defining identity for Hom-associative algebras [14]. When $n = 3$ total Hom-associativity means

$$((a_1 a_2 a_3), \alpha_1(a_4), \alpha_2(a_5)) = (\alpha_1(a_1), (a_2 a_3 a_4), \alpha_2(a_5)) = (\alpha_1(a_1), \alpha_2(a_2), (a_3 a_4 a_5)). \quad (2.3.1)$$

In particular, if both twisting maps α_1 and α_2 are equal to the identity map, then (2.3.1) is the defining identity for ternary rings [12].

To see that the category of n -ary totally Hom-associative algebras is closed under twisting by self-weak morphisms, we need the following observations.

Lemma 2.4. *Let $(A, (, \dots), \alpha)$ be an n -ary Hom-algebra and $\beta: A \rightarrow A$ be a weak morphism. Consider the n -ary Hom-algebra*

$$A_\beta = (A, (, \dots)_\beta = \beta(, \dots), \beta\alpha = (\beta\alpha_1, \dots, \beta\alpha_{n-1})). \quad (2.4.1)$$

Then the following statements hold.

- (1) $\beta^2 a s_A^i = a s_{A_\beta}^i$ for $i \in \{1, \dots, n-1\}$.
- (2) *If A is multiplicative and $\beta\alpha = \alpha\beta$, then A_β is also multiplicative.*

Proof. Both assertions are immediate from the definitions. □

The desired closure property is now an immediate consequence of Lemma 2.4.

Theorem 2.5. *Let $(A, (, \dots), \alpha)$ be an n -ary totally Hom-associative algebra and $\beta: A \rightarrow A$ be a weak morphism. Then A_β in (2.4.1) is also an n -ary totally Hom-associative algebra. Moreover, if A is multiplicative and $\beta\alpha = \alpha\beta$, then A_β is also multiplicative.*

Let us now discuss some special cases of Theorem 2.5. If A is a multiplicative n -ary Hom-algebra, then its twisting map is a morphism (and hence a weak morphism) on A . Therefore, we have the following special case of Theorem 2.5.

Corollary 2.6. *Let $(A, (, \dots), \alpha)$ be a multiplicative n -ary totally Hom-associative algebra. Then*

$$A_\alpha = (A, (, \dots)_\alpha = \alpha(, \dots), \alpha^2)$$

is also a multiplicative n -ary totally Hom-associative algebra.

Iterating Corollary 2.6 we obtain the following result, which says that every multiplicative n -ary totally Hom-associative algebra gives rise to a sequence of derived n -ary totally Hom-associative algebras.

Corollary 2.7. *Let $(A, (, \dots), \alpha)$ be a multiplicative n -ary totally Hom-associative algebra. Then*

$$A_k = (A, (, \dots)_k = \alpha^{2^k-1}(, \dots), \alpha^{2^k})$$

is also a multiplicative n -ary totally Hom-associative algebra for each $k \geq 0$.

On the other hand, if we set $\alpha_i = Id$ for all i in Theorem 2.5, then we obtain the following twisting result, which is Theorem 3.6 in [3]. It says that n -ary totally Hom-associative algebras can be obtained from n -ary totally associative algebras and their morphisms.

Corollary 2.8. *Let $(A, (, \dots,))$ be an n -ary totally associative algebra and $\beta: A \rightarrow A$ be a morphism. Then*

$$A_\beta = (A, (, \dots,)_\beta = \beta(, \dots,), \beta)$$

is a multiplicative n -ary totally Hom-associative algebra.

The rest of this section contains examples of n -ary totally Hom-associative algebras.

Example 2.9. Let A be an associative algebra, $f: A \rightarrow A$ be an algebra morphism, and $\zeta \in \mathbf{k}$ be a primitive n th root of unity. Then the ζ -eigenspace of f ,

$$A(f, \zeta) = \{a \in A: f(a) = \zeta a\}, \quad (2.9.1)$$

is an $(n+1)$ -ary totally associative algebra under the $(n+1)$ -ary product

$$(a_1, \dots, a_{n+1}) = a_1 \cdots a_{n+1}. \quad (2.9.2)$$

Let $\alpha: A \rightarrow A$ be an algebra morphism such that $\alpha f = f\alpha$. Then α restricts to a morphism of $(n+1)$ -ary totally associative algebras on $A(f, \zeta)$. By Corollary 2.8 there is a multiplicative $(n+1)$ -ary totally Hom-associative algebra

$$A(f, \zeta)_\alpha = (A(f, \zeta), (, \dots,)_\alpha, \alpha),$$

where

$$(a_1, \dots, a_{n+1})_\alpha = \alpha(a_1 \cdots a_{n+1})$$

for all $a_i \in A$. □

Example 2.10. Let A be the associative algebra over \mathbf{k} consisting of polynomials in $r \geq 2$ associative variables X_1, \dots, X_r with 0 constant term. Fix an integer $n \geq 2$. Let A_n denote the submodule of A spanned by the homogeneous polynomials of degrees $1 \pmod{n}$. Note that A_n is actually an eigenspace $A(f, \zeta)$ as in (2.9.1), where $\zeta \in \mathbf{k}$ is a primitive n th root of unity and $f: A \rightarrow A$ is determined by

$$f(X_i) = \zeta X_i$$

for all i . In any case, A_n is an $(n+1)$ -ary totally associative algebra with the $(n+1)$ -ary product in (2.9.2). When $n = 2$ the ternary totally associative algebra A_3 is an example in [12] (p.47).

For each $i \in \{1, \dots, r\}$ let $m_i \geq 1$ be an integer with $m_i \equiv 1 \pmod{n}$. Then the map $\alpha: A \rightarrow A$ determined by

$$\alpha(X_i) = X_i^{m_i}$$

for all i is an algebra morphism that commutes with f . As in Example 2.9 there is a multiplicative $(n+1)$ -ary totally Hom-associative algebra $(A_n)_\alpha = (A_n, (, \dots,)_\alpha, \alpha)$.

Note that $(A_n, (, \dots,)_\alpha)$ is not totally associative because

$$((\underbrace{X_1 \cdots X_1}_{n+1})_\alpha \underbrace{X_2 \cdots X_2}_n)_\alpha = X_1^{m_1^2(n+1)} X_2^{m_2 n},$$

whereas

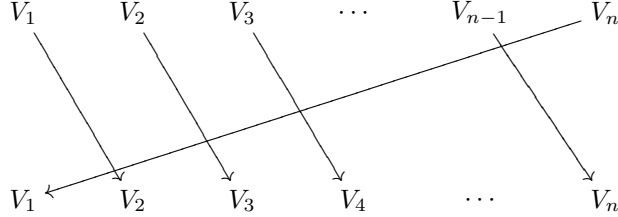
$$(\underbrace{X_1 \cdots X_1}_n (X_1 \underbrace{X_2 \cdots X_2}_n)_\alpha)_\alpha = X_1^{m_1(m_1+n)} X_2^{m_2^2 n}.$$

They are not equal, provided $m_1 > 1$ or $m_2 > 1$. □

Example 2.11. Fix an integer $n \geq 2$, and let V_1, \dots, V_n be \mathbf{k} -modules. Consider the direct sum

$$A = \bigoplus_{i=1}^n \text{Hom}(V_i, V_{i+1}),$$

where $V_{n+1} \equiv V_1$. A typical element in A is written as $\oplus f_i$, where $f_i \in \text{Hom}(V_i, V_{i+1})$ for $i \in \{1, \dots, n\}$. One can visualize the element $\oplus f_i \in A$ as the braid diagram



on n strands. Define an $(n+1)$ -ary product on A by

$$(\oplus f_i^1, \dots, \oplus f_i^{n+1}) = \oplus F_i,$$

where

$$F_i = \underbrace{f_i^{n+1} \dots f_1^{n-i+2}}_i \underbrace{f_n^{n-i+1} \dots f_{i+1}^2 f_i^1}_{n-i+1}$$

for each i . Pictorially, the $(n+1)$ -ary product $\oplus F_i$ is represented by the vertical composition of $n+1$ braid diagrams. With this $(n+1)$ -ary product, A becomes an $(n+1)$ -ary totally associative algebra. When $n=2$ the ternary totally associative algebra A is an example in [12] (p.46).

For each $i \in \{1, \dots, n\}$, let $\gamma_i: V_i \rightarrow V_i$ be a linear automorphism. Define the map $\alpha: A \rightarrow A$ by

$$\alpha(\oplus f_i) = \oplus \gamma_{i+1}^{-1} f_i \gamma_i,$$

where $\gamma_{n+1} \equiv \gamma_1$. Then α is an automorphism of $(n+1)$ -ary totally associative algebras. By Corollary 2.8 there is a multiplicative n -ary totally Hom-associative algebra

$$A_\alpha = (A, (\dots)_\alpha = \alpha(\dots), \alpha).$$

Moreover, $(A, (\dots)_\alpha)$ is in general not an n -ary totally associative algebra. For example, suppose $f_i^i \in \text{Hom}(V_i, V_{i+1})$ for $i \in \{1, \dots, n\}$ and $f_1^{n+1} \in \text{Hom}(V_1, V_2)$. Then

$$((f_1^1, \dots, f_n^n, f_1^{n+1})_\alpha, f_2^2, \dots, f_1^{n+1})_\alpha = \gamma_2^{-1} f_1^{n+1} \dots f_2^2 \gamma_2^{-1} F_1 \gamma_1^2,$$

whereas

$$(f_1^1, \dots, f_n^n, (f_1^{n+1}, f_2^2, \dots, f_1^{n+1})_\alpha)_\alpha = \gamma_2^{-2} f_1^{n+1} \dots f_2^2 f_1^{n+1} \gamma_1 f_n^n \dots f_1^1 \gamma_1.$$

They are not equal in general. \square

3. TOTALLY HOM-ASSOCIATIVE ALGEBRAS OF DIFFERENT ARITIES

There are two main results in this section. The first main result (Theorem 3.1) says that every multiplicative n -ary totally Hom-associative algebra yields a multiplicative $(2n-1)$ -ary totally Hom-associative algebra. The second main result (Theorem 3.4) says that under suitable conditions an n -ary totally Hom-associative algebra reduces to an $(n-1)$ -ary totally Hom-associative algebra. Both of these results can be iterated.

Here is the first main result of this section. The Hom-Nambu analogue is discussed in [24]. Recall the abbreviations in (2.1.1).

Theorem 3.1. *Let $(A, (\dots), \alpha)$ be a multiplicative n -ary totally Hom-associative algebra. Then*

$$A^1 = (A, (\dots)^{(1)}, \alpha^2)$$

is a multiplicative $(2n-1)$ -ary totally Hom-associative algebra, where

$$(a_{1,2n-1})^{(1)} = ((a_{1,n}), \alpha(a_{n+1,2n-1}))$$

for all $a_i \in A$.

Proof. It is clear that A^1 is a multiplicative $(2n-1)$ -ary Hom-algebra. We must show that its j th Hom-associator $as_{A^1}^j$ (Definition 2.3) is equal to 0, that is,

$$\begin{aligned} & \left(\alpha^2(a_{1,j-1}), (a_{j,j+2n-2})^{(1)}, \alpha^2(a_{j+2n-1,4n-3}) \right)^{(1)} \\ &= \left(\alpha^2(a_{1,j}), (a_{j+1,j+2n-1})^{(1)}, \alpha^2(a_{j+2n,4n-3}) \right)^{(1)} \end{aligned} \quad (3.1.1)$$

for all $j \in \{1, \dots, 2n-2\}$. The condition (3.1.1) is divided into three cases: (1) $j \leq n-1$, (2) $j = n$, and (3) $j \geq n+1$. Note that the case $j \geq n+1$ does not occur if $n = 2$. The three cases are proved similarly, so we only provide the details for case (3), which is the only case where multiplicativity is used. With $j \geq n+1$, using the multiplicativity and total Hom-associativity of A , we compute the left-hand side of (3.1.1) as follows:

$$\begin{aligned} & \left(\alpha^2(a_{1,j-1}), (a_{j,j+2n-2})^{(1)}, \alpha^2(a_{j+2n-1,4n-3}) \right)^{(1)} \\ &= ((\alpha^2(a_{1,n})), \alpha^3(a_{n+1,j-1}), \alpha((a_{j,j+n-1}), \alpha(a_{j+n,j+2n-2})), \alpha^3(a_{j+2n-1,4n-3})) \\ &= ((\alpha^2(a_{1,n})), \alpha^3(a_{n+1,j-1}), \alpha(\alpha(a_j), (a_{j+1,j+n}), \alpha(a_{j+n+1,j+2n-2})), \alpha^3(a_{j+2n-1,4n-3})) \\ &= (\alpha^2((a_{1,n})), \alpha^3(a_{n+1,j-1}), (\alpha^2(a_j), \alpha(a_{j+1,j+n}), \alpha^2(a_{j+n+1,j+2n-2})), \alpha^3(a_{j+2n-1,4n-3})) \\ &= (\alpha^2((a_{1,n})), \alpha^3(a_{n+1,j}), (\alpha(a_{j+1,j+n}), \alpha^2(a_{j+n+1,j+2n-1})), \alpha^3(a_{j+2n,4n-3})) \\ &= ((\alpha^2(a_{1,n})), \alpha^3(a_{n+1,j}), \alpha((a_{j+1,j+n}), \alpha(a_{j+n+1,j+2n-1})), \alpha^3(a_{j+2n,4n-3})) \end{aligned}$$

The last expression above is equal to the right-hand side of (3.1.1), as desired. \square

Applying Theorem 3.1 repeatedly, we obtain the following result. It says that every multiplicative n -ary totally Hom-associative algebra gives rise to a sequence of multiplicative totally Hom-associative algebras of exponentially higher arities.

Corollary 3.2. *Let $(A, (\dots), \alpha)$ be a multiplicative n -ary totally Hom-associative algebra. Define the $(2^k(n-1)+1)$ -ary product $(\dots)^{(k)}$ inductively by setting $(\dots)^{(0)} = (\dots)$ and*

$$(a_{1,2^k(n-1)+1})^{(k)} = \left((a_{1,2^{k-1}(n-1)+1})^{(k-1)}, \alpha^{2^{k-1}}(a_{2^{k-1}(n-1)+2, 2^k(n-1)+1}) \right)^{(k-1)}$$

for $k \geq 1$. Then

$$A^k = (A, (\dots)^{(k)}, \alpha^{2^k})$$

is a multiplicative $(2^k(n-1)+1)$ -ary totally Hom-associative algebra for each $k \geq 0$.

For example, when $k = 2$ we have:

$$\begin{aligned} (a_{1,4n-3})^{(2)} &= \left((a_{1,2n-1})^{(1)}, \alpha^2(a_{2n,4n-3}) \right)^{(1)} \\ &= (((a_{1,n}), \alpha(a_{n+1,2n-1})), \alpha^2(a_{2n,3n-2})), \alpha^3(a_{3n-1,4n-3}) \end{aligned}$$

When $k = 3$, writing $x = (a_{1,4n-3})^{(2)}$, we have:

$$\begin{aligned} (a_{1,8n-7})^{(3)} &= (x, \alpha^4(a_{4n-2,8n-7}))^{(2)} \\ &= (((x, \alpha^4(a_{4n-2,5n-4})), \alpha^5(a_{5n-3,6n-5})), \alpha^6(a_{6n-4,7n-6})), \alpha^7(a_{7n-5,8n-7})). \end{aligned}$$

In general, $(, \dots)^{(k)}$ involves an iterated composition of 2^k copies of the n -ary product $(, \dots)$ and $n - 1$ copies of α^i for each $i \in \{1, \dots, 2^k - 1\}$.

Restricting to the case $\alpha = Id$ in Corollary 3.2, we obtain the following construction result for higher arity totally associative algebras.

Corollary 3.3. *Let $(A, (, \dots))$ be an n -ary totally associative algebra. Define the $(2^k(n-1)+1)$ -ary product $(, \dots)^{(k)}$ inductively by setting $(, \dots)^{(0)} = (, \dots)$ and*

$$(a_{1,2^k(n-1)+1})^{(k)} = \left((a_{1,2^{k-1}(n-1)+1})^{(k-1)}, a_{2^{k-1}(n-1)+2,2^k(n-1)+1} \right)^{(k-1)}$$

for $k \geq 1$. Then

$$A^k = (A, (, \dots)^{(k)})$$

is a multiplicative $(2^k(n-1)+1)$ -ary totally associative algebra for each $k \geq 0$.

The following result is the second main result of this section. It gives sufficient conditions under which an n -ary totally Hom-associative algebra reduces to an $(n-1)$ -ary totally Hom-associative algebra. This result is the totally Hom-associative analogue of results due to Pozhidaev [19] and Filippov [8] about n -ary Maltsev algebras and Nambu-Lie algebras, respectively. The n -ary Hom-Nambu(-Lie) and Hom-Maltsev analogues can be found in [24] and [25], respectively.

Theorem 3.4. *Let $(A, (, \dots), \alpha)$ be an n -ary totally Hom-associative algebra with $n \geq 3$. Suppose $a \in A$ satisfies*

- (1) $\alpha_{n-1}(a) = a$, and
- (2) $(x_{1,n-1}, a) = (x_{1,n-2}, a, x_{n-1})$ for all $x_i \in A$.

Then

$$A_1 = (A, (, \dots)', \alpha')$$

is an $(n-1)$ -ary totally Hom-associative algebra, where

$$(x_{1,n-1})' = (x_{1,n-1}, a) \quad \text{and} \quad \alpha' = (\alpha_1, \dots, \alpha_{n-2})$$

for all $x_i \in A$. Moreover, if A is multiplicative, then so is A_1 .

Proof. The multiplicativity assertion is clear. We must show that for $j \in \{1, \dots, n-2\}$, the j th Hom-associator $as_{A_1}^j$ is equal to 0, that is,

$$\begin{aligned} &(\alpha_{1,j-1}(x_{1,j-1}), (x_{j,j+n-2})', \alpha_{j,n-2}(x_{j+n-1,2n-3}))' \\ &= (\alpha_{1,j}(x_{1,j}), (x_{j+1,j+n-1})', \alpha_{j+1,n-2}(x_{j+n,2n-3}))'. \end{aligned} \tag{3.4.1}$$

Using the assumptions on a and the total Hom-associativity of A , we compute the left-hand side of (3.4.1) as follows:

$$\begin{aligned} &(\alpha_{1,j-1}(x_{1,j-1}), (x_{j,j+n-2})', \alpha_{j,n-2}(x_{j+n-1,2n-3}))' \\ &= (\alpha_{1,j-1}(x_{1,j-1}), (x_{j,j+n-2}, a), \alpha_{j,n-2}(x_{j+n-1,2n-3}), \alpha_{n-1}(a)) \\ &= (\alpha_{1,j}(x_{1,j}), (x_{j+1,j+n-2}, a, x_{j+n-1}), \alpha_{j+1,n-2}(x_{j+n,2n-3}), \alpha_{n-1}(a)) \\ &= (\alpha_{1,j}(x_{1,j}), (x_{j+1,j+n-1}, a), \alpha_{j+1,n-2}(x_{j+n,2n-3}), a). \end{aligned}$$

The last expression above is equal to the right-hand side of (3.4.1), as desired. \square

Note that in Theorem 3.4 the first assumption about a is automatically satisfied if α_{n-1} is the identity map on A . On the other hand, the second assumption on a is automatically satisfied if the n -ary product $(, \dots,)$ is commutative in the last two variables.

Applying Theorem 3.4 repeatedly, we obtain the following result. It gives sufficient conditions under which an n -ary totally Hom-associative algebra reduces to an $(n-k)$ -ary totally Hom-associative algebra.

Corollary 3.5. *Let $(A, (, \dots,), \alpha)$ be an n -ary totally Hom-associative algebra with $n \geq 3$. Suppose there exist $a_1, \dots, a_k \in A$ for some $k \leq n-2$ such that*

- (1) $\alpha_{n-i}(a_i) = a_i$ for all $i \in \{1, \dots, k\}$, and
- (2) $(x_{1,n-i}, a_i, a_{i-1}, \dots, a_1) = (x_{1,n-i-1}, a_i, x_{n-i}, a_{i-1}, \dots, a_1)$ for all $i \in \{1, \dots, k\}$ and $x_j \in A$.

Then

$$A_k = (A, (, \dots,)^k, (\alpha_1, \dots, \alpha_{n-1-k}))$$

is an $(n-k)$ -ary totally Hom-associative algebra, where

$$(x_{1,n-k})^k = (x_{1,n-k}, a_k, a_{k-1}, \dots, a_1)$$

for all $x_j \in A$. Moreover, if A is multiplicative, then so is A_k .

4. FROM TOTALLY HOM-ASSOCIATIVE ALGEBRAS TO HOM-NAMBU ALGEBRAS

The main result of this section says that an n -ary totally Hom-associative algebra with equal twisting maps yields an n -ary Hom-Nambu algebra via the n -commutator bracket.

Let us first define the n -commutator words, which generalize the two terms in the usual commutator bracket.

Definition 4.1. Let X_1, X_2, \dots be non-commuting variables. For $n \geq 2$ define the set W_n of n -commutator words inductively as

$$W_2 = \{X_1X_2, -X_2X_1\}$$

and

$$W_n = \{zX_n, -X_nz : z \in W_{n-1}\}$$

for $n > 2$.

For example,

$$W_3 = \{X_1X_2X_3, -X_2X_1X_3, -X_3X_1X_2, X_3X_2X_1\}$$

and W_4 consists of the 4-commutator words

$$\begin{aligned} &X_1X_2X_3X_4, -X_2X_1X_3X_4, -X_3X_1X_2X_4, X_3X_2X_1X_4, \\ &-X_4X_1X_2X_3, X_4X_2X_1X_3, X_4X_3X_1X_2, -X_4X_3X_2X_1. \end{aligned}$$

In general, W_n consists of 2^{n-1} n -commutator words. Every n -commutator word gives a self-map on the n -fold tensor product as follows.

Definition 4.2. Let V be a \mathbf{k} -module and $w = \pm X_{i_1} \cdots X_{i_n} \in W_n$. Define the map $w: V^{\otimes n} \rightarrow V^{\otimes n}$ by

$$w(v_1, \dots, v_n) = \pm v_{i_1} \otimes \cdots \otimes v_{i_n}$$

for all $v_j \in V$.

Using these maps defined by the n -commutator words, we can now define the n -commutator bracket.

Definition 4.3. Let $(A, (\dots), \alpha)$ be an n -ary Hom-algebra. Define the n -commutator bracket $[\dots]: A^{\otimes n} \rightarrow A^{\otimes n}$ by

$$[a_1, \dots, a_n] = \sum_{w \in W_n} (w(a_1, \dots, a_n)) \quad (4.3.1)$$

for all $a_j \in A$.

For example, the 2-commutator bracket is the usual commutator bracket:

$$[a_1, a_2] = (a_1 a_2) - (a_2 a_1).$$

The 3-commutator bracket is the sum

$$[a_{1,3}] = (a_1 a_2 a_3) - (a_2 a_1 a_3) - (a_3 a_1 a_2) + (a_3 a_2 a_1),$$

which was first used in Corollary 4.3 in [24].

Let us now recall the definition of an n -ary Hom-Nambu algebra from [3].

Definition 4.4. Let $(V, [\dots], \alpha)$ be an n -ary Hom-algebra.

- (1) The n -ary **Hom-Jacobian** of V is the $(2n-1)$ -linear map $J_V^n: V^{\otimes 2n-1} \rightarrow V$ defined as (using the shorthand in (2.1.1))

$$\begin{aligned} J_V^n(x_{1,n-1}; y_{1,n}) &= [\alpha_{1,n-1}(x_{1,n-1}), [y_{1,n}]] \\ &\quad - \sum_{i=1}^n [\alpha_{1,i-1}(y_{1,i-1}), [x_{1,n-1}, y_i], \alpha_{i,n-1}(y_{i+1,n})] \end{aligned} \quad (4.4.1)$$

for $x_1, \dots, x_{n-1}, y_1, \dots, y_n \in V$.

- (2) An n -ary **Hom-Nambu algebra** is an n -ary Hom-algebra V that satisfies

$$J_V^n = 0,$$

called the n -ary **Hom-Nambu identity**.

When the twisting maps are all equal to the identity map, n -ary Hom-Nambu algebras are the usual n -ary **Nambu algebras** [4, 8, 16, 18]. In this case, the n -ary Hom-Jacobian J_V^n is called the n -ary **Jacobian**, and the n -ary Hom-Nambu identity $J_V^n = 0$ is called the n -ary **Nambu identity**.

We are now ready for the main result of this section.

Theorem 4.5. Let $(A, (\dots), \alpha)$ be an n -ary totally Hom-associative algebra with equal twisting maps. Then

$$N(A) = (A, [\dots], \alpha)$$

is an n -ary Hom-Nambu algebra, where $[\dots]$ is the n -commutator bracket in (4.3.1). Moreover, if A is multiplicative, then so is $N(A)$.

Since the n -commutator bracket does not involve the twisting maps, we have the following special case of Theorem 4.5

Corollary 4.6. *Let $(A, (\dots))$ be an n -ary totally associative algebra with equal twisting maps. Then*

$$N(A) = (A, [\dots])$$

is an n -ary Nambu algebra, where $[\dots]$ is the n -commutator bracket.

For the proof of Theorem 4.5, we need to prove the n -ary Hom-Nambu identity $J_{N(A)}^n = 0$. In the following Lemmas, we first compute the various terms in the n -ary Hom-Jacobian $J_{N(A)}^n$ (4.4.1). The n terms in the sum in (4.4.1) are considered in two cases, $i = n$ (Lemma 4.8) and $1 \leq i \leq n-1$ (Lemma 4.10).

Let us compute the first term in the n -ary Hom-Jacobian $J_{N(A)}^n$.

Lemma 4.7. *With the hypotheses of Theorem 4.5, we have*

$$\begin{aligned} [\alpha(x_{1,n-1}), [y_{1,n}]] &= \sum_{z, z' \in W_{n-1}} (z'(\alpha(x_{1,n-1})), (z(y_{1,n-1}), y_n)) \\ &\quad - \sum_{z, z' \in W_{n-1}} (z'(\alpha(x_{1,n-1})), (y_n, z(y_{1,n-1}))) \\ &\quad - \sum_{z, z' \in W_{n-1}} ((z(y_{1,n-1}), y_n), z'(\alpha(x_{1,n-1}))) \\ &\quad + \sum_{z, z' \in W_{n-1}} ((y_n, z(y_{1,n-1})), z'(\alpha(x_{1,n-1}))) \end{aligned} \tag{4.7.1}$$

for all $x_j, y_l \in A$.

Proof. From the definitions of the n -commutator bracket (4.3.1) and the n -commutator words (Definition 4.1), we have

$$\begin{aligned} [a_{1,n}] &= \sum_{w \in W_n} (w(a_{1,n})) \\ &= \sum_{z \in W_{n-1}} (z(a_{1,n-1}), a_n) - (a_n, z(a_{1,n-1})). \end{aligned} \tag{4.7.2}$$

for all $a_j \in A$. The Lemma is obtained by using (4.7.2) on both $[y_{1,n}]$ and $[\alpha(x_{1,n-1}), \cdot]$. \square

In Lemma 4.7 we did not use the total Hom-associativity of A . In the next result, we compute the $i = n$ term in the sum in the n -ary Hom-Jacobian $J_{N(A)}^n$.

Lemma 4.8. *With the hypotheses of Theorem 4.5, we have*

$$\begin{aligned} -[\alpha(y_{1,n-1}), [x_{1,n-1}, y_n]] &= - \sum_{z, z' \in W_{n-1}} (z(\alpha(y_{1,n-1})), (z'(x_{1,n-1}), y_n)) \\ &\quad + \sum_{z, z' \in W_{n-1}} ((z(y_{1,n-1}), y_n), z'(\alpha(x_{1,n-1}))) \\ &\quad + \sum_{z, z' \in W_{n-1}} (z'(\alpha(x_{1,n-1})), (y_n, z(y_{1,n-1}))) \\ &\quad - \sum_{z, z' \in W_{n-1}} ((y_n, z'(x_{1,n-1})), z(\alpha(y_{1,n-1}))) \end{aligned} \tag{4.8.1}$$

for all $x_j, y_l \in A$.

Proof. Using (4.7.2) we have

$$\begin{aligned}
-[\alpha(y_{1,n-1}), [x_{1,n-1}, y_n]] &= - \sum_{z, z' \in W_{n-1}} (z(\alpha(y_{1,n-1})), (z'(x_{1,n-1}), y_n)) \\
&+ \sum_{z, z' \in W_{n-1}} (z(\alpha(y_{1,n-1})), (y_n, z'(x_{1,n-1}))) \\
&+ \sum_{z, z' \in W_{n-1}} ((z'(x_{1,n-1}), y_n), z(\alpha(y_{1,n-1}))) \\
&- \sum_{z, z' \in W_{n-1}} ((y_n, z'(x_{1,n-1})), z(\alpha(y_{1,n-1}))). \tag{4.8.2}
\end{aligned}$$

The first (resp., fourth) sums in (4.8.1) and (4.8.2) are equal. The second (resp., third) sums in (4.8.1) and (4.8.2) are equal by the total Hom-associativity of A . \square

Combining Lemmas 4.7 and 4.8, we obtain the following result.

Lemma 4.9. *With the hypotheses of Theorem 4.5, we have*

$$\begin{aligned}
&[\alpha(x_{1,n-1}), [y_{1,n}]] - [\alpha(y_{1,n-1}), [x_{1,n-1}, y_n]] \\
&= \sum_{z, z' \in W_{n-1}} \underbrace{(z'(\alpha(x_{1,n-1})), (z(y_{1,n-1}), y_n))}_{A_1} \\
&+ \sum_{z, z' \in W_{n-1}} \underbrace{((y_n, z(y_{1,n-1})), z'(\alpha(x_{1,n-1})))}_{A_4} \\
&- \sum_{z, z' \in W_{n-1}} \underbrace{(z(\alpha(y_{1,n-1})), (z'(x_{1,n-1}), y_n))}_{B_{n1}} \\
&- \sum_{z, z' \in W_{n-1}} \underbrace{((y_n, z'(x_{1,n-1})), z(\alpha(y_{1,n-1})))}_{B_{n4}}
\end{aligned}$$

for all $x_j, y_l \in A$.

Proof. This follows from Lemmas 4.7 and 4.8 because the second (resp., third) sum in (4.7.1) is equal to the third (resp., second) sum in (4.8.1) with the opposite sign. \square

The labels A_1 , A_4 , B_{n1} , and B_{n4} will be used below.

Next we compute the i th term ($1 \leq i \leq n-1$) in the sum in the n -ary Hom-Jacobian $J_{N(A)}^n$.

Lemma 4.10. *With the hypotheses of Theorem 4.5, for $i \in \{1, \dots, n-1\}$ we have*

$$\begin{aligned}
& -[\alpha(y_{1,i-1}), [x_{1,n-1}, y_i], \alpha(y_{i+1,n})] \\
&= - \sum_{z, z' \in W_{n-1}} \underbrace{(z(\alpha(y_{1,i-1})), (z'(x_{1,n-1}), y_i), \alpha(y_{i+1,n-1})), \alpha(y_n))}_{B_{i1}} \\
&+ \sum_{z, z' \in W_{n-1}} \underbrace{(\alpha(y_n), z(\alpha(y_{1,i-1})), (z'(x_{1,n-1}), y_i), \alpha(y_{i+1,n-1})))}_{B_{i2}} \\
&+ \sum_{z, z' \in W_{n-1}} \underbrace{(z(\alpha(y_{1,i-1})), (y_i, z'(x_{1,n-1})), \alpha(y_{i+1,n-1})), \alpha(y_n))}_{B_{i3}} \\
&- \sum_{z, z' \in W_{n-1}} \underbrace{(\alpha(y_n), z(\alpha(y_{1,i-1})), (y_i, z'(x_{1,n-1})), \alpha(y_{i+1,n-1})))}_{B_{i4}}
\end{aligned}$$

for all $x_j, y_l \in A$.

Proof. Just use (4.7.2) twice, as in Lemma 4.7. \square

Using Lemmas 4.9 and 4.10 we can now give the proof of Theorem 4.5.

Proof of Theorem 4.5. Since the multiplicativity assertion is clear, it remains to establish the n -ary Hom-Nambu identity $J_{N(A)}^n = 0$. By Lemmas 4.9 and 4.10 the n -ary Hom-Jacobian $J_{N(A)}^n$ is

$$J_{N(A)}^n = \sum_{z, z' \in W_{n-1}} \left(A_1 + A_4 - B_{n1} - B_{n4} + \sum_{i=1}^{n-1} (-B_{i1} + B_{i2} + B_{i3} - B_{i4}) \right). \quad (4.10.1)$$

The total Hom-associativity of A implies the following six types of cancellation. For $i \in \{1, \dots, n-1\}$ we have:

$$\begin{aligned} \sum_{\substack{z' \in W_{n-1} \\ z = \pm X_i \cdots \in W_{n-1}}} (A_1 - B_{i1}) &= 0, & \sum_{\substack{z' \in W_{n-1} \\ z = \pm \cdots X_i \in W_{n-1}}} (A_4 - B_{i4}) &= 0, \\ \sum_{\substack{z' \in W_{n-1} \\ z = \pm \cdots X_i \in W_{n-1}}} (-B_{n1} + B_{i3}) &= 0, & \sum_{\substack{z' \in W_{n-1} \\ z = \pm X_i \cdots \in W_{n-1}}} (-B_{n4} + B_{i2}) &= 0. \end{aligned}$$

For $i \neq j$ in $\{1, \dots, n-1\}$ we have:

$$\sum_{\substack{z' \in W_{n-1} \\ z = \pm \cdots X_j X_i \cdots \in W_{n-1}}} (-B_{i1} + B_{j3}) = 0, \quad \sum_{\substack{z' \in W_{n-1} \\ z = \pm \cdots X_j X_i \cdots \in W_{n-1}}} (B_{i2} - B_{j4}) = 0.$$

Here $z = \pm X_i \cdots \in W_{n-1}$ (resp., $z = \pm \cdots X_i \in W_{n-1}$) means that z is an $(n-1)$ -commutator word starting (resp., ending) with X_i . Likewise, $z = \pm \cdots X_j X_i \cdots \in W_{n-1}$ means that z is an $(n-1)$ -commutator word in which X_i is immediately preceded by X_j . Using the expression (4.10.1) for $J_{N(A)}^n$, these six types of cancellation imply that $N(A)$ satisfies the n -ary Hom-Nambu identity $J_{N(A)}^n = 0$. \square

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